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Research Article

Positive Solutions of a Nonlinear Three-Point Integral Boundary Value Problem

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We study the existence of positive solutions to the three-point integral boundary value problem $u'' + a(t)f(u) = 0$, $t \in (0,1)$, $u(0) = 0$, $\alpha \int_0^\eta u(s)ds = u(1)$, where $0 < \eta < 1$ and $0 < \alpha < 2/\eta^2$. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.

1. Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by several authors. We refer the reader to [3–19] and the references therein. However, all these papers are concerned with problems with three-point boundary condition restrictions on the slope of the solutions and the solutions themselves, for example,

$$\begin{aligned}
u(0) &= 0, & \alpha u(\eta) &= u(1),\\
u(0) &= \beta u(\eta), & \alpha u(\eta) &= u(1),\\
u'(0) &= 0, & \alpha u(\eta) &= u(1),\\
u(0) - \beta u'(0) &= 0, & \alpha u(\eta) &= u(1),\\
\alpha u(0) - \beta u'(0) &= 0, & u'(\eta) + u'(1) &= 0,
\end{aligned} \tag{1.1}$$

and so forth.

In this paper, we consider the existence of positive solutions to the equation

$$u'' + a(t)f(u) = 0, \quad t \in (0,1), \tag{1.2}$$

with the three-point integral boundary condition

$$u(0) = 0, \quad \alpha \int_0^\eta u(s)ds = u(1), \tag{1.3}$$

where $0 < \eta < 1$. We note that the new three-point boundary conditions are related to the area under the curve of solutions $u(t)$ from $t = 0$ to $t = \eta$.

The aim of this paper is to give some results for existence of positive solutions to (1.2)-(1.3), assuming that $0 < \alpha < 2/\eta^2$ and f is either superlinear or sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}. \tag{1.4}$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. By the positive solution of (1.2)-(1.3) we mean that a function $u(t)$ is positive on $0 < t < 1$ and satisfies the problem (1.2)-(1.3).

Throughout this paper, we suppose the following conditions hold:

$(H1)$ $f \in C([0,\infty),[0,\infty))$;

$(H2)$ $a \in C([0,1],[0,\infty))$ and there exists $t_0 \in [\eta, 1]$ such that $a(t_0) > 0$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1 (see [20]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega}_1 \setminus \Omega_2\right) \longrightarrow K \tag{1.5}$$

be a completely continuous operator such that

(i) $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *Let $\alpha\eta^2 \neq 2$. Then for $y \in C[0,1]$, the problem*

$$u'' + y(t) = 0, \quad t \in (0,1), \tag{2.1}$$

$$u(0) = 0, \qquad \alpha \int_0^\eta u(s)ds = u(1), \tag{2.2}$$

has a unique solution

$$u(t) = \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)y(s)ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta - s)^2 y(s)ds - \int_0^t (t-s)y(s)ds. \tag{2.3}$$

Proof. From (2.1), we have

$$u''(t) = -y(t). \tag{2.4}$$

For $t \in [0,1)$, integration from 0 to t, gives

$$u'(t) = u'(0) - \int_0^t y(s)ds. \tag{2.5}$$

For $t \in [0,1]$, integration from 0 to t yields that

$$u(t) = u'(0)t - \int_0^t \left(\int_0^x y(s)ds\right)dx, \tag{2.6}$$

that is,

$$u(t) = u'(0)t - \int_0^t (t-s)y(s)ds. \tag{2.7}$$

So,

$$u(1) = u'(0) - \int_0^1 (1-s)y(s)ds. \tag{2.8}$$

Integrating (2.7) from 0 to η, where $\eta \in (0,1)$, we have

$$\begin{aligned}\int_0^\eta u(s)ds &= u'(0)\frac{\eta^2}{2} - \int_0^\eta\left(\int_0^x (x-s)y(s)ds\right)dx \\ &= u'(0)\frac{\eta^2}{2} - \frac{1}{2}\int_0^\eta (\eta-s)^2 y(s)ds.\end{aligned} \tag{2.9}$$

From (2.2), we obtain that

$$u'(0) - \int_0^1 (1-s)y(s)ds = u'(0)\frac{\alpha\eta^2}{2} - \frac{\alpha}{2}\int_0^\eta (\eta-s)^2 y(s)ds. \tag{2.10}$$

Thus,

$$u'(0) = \frac{2}{2-\alpha\eta^2}\int_0^1 (1-s)y(s)ds - \frac{\alpha}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 y(s)ds. \tag{2.11}$$

Therefore, (2.1)-(2.2) has a unique solution

$$u(t) = \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)y(s)ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 y(s)ds - \int_0^t (t-s)y(s)ds. \tag{2.12}$$

□

Lemma 2.2. *Let $0 < \alpha < 2/\eta^2$. If $y \in C(0,1)$ and $y(t) \geqslant 0$ on $(0,1)$, then the unique solution u of (2.1)-(2.2) satisfies $u \geqslant 0$ for $t \in [0,1]$.*

Proof. If $u(1) \geqslant 0$, then, by the concavity of u and the fact that $u(0) = 0$, we have $u(t) \geqslant 0$ for $t \in [0,1]$.

Moreover, we know that the graph of $u(t)$ is concave down on $(0,1)$, we get

$$\int_0^\eta u(s)ds \geqslant \frac{1}{2}\eta u(\eta), \tag{2.13}$$

where $(1/2)\eta u(\eta)$ is the area of triangle under the curve $u(t)$ from $t = 0$ to $t = \eta$ for $\eta \in (0,1)$.

Assume that $u(1) < 0$. From (2.2), we have

$$\int_0^\eta u(s)ds < 0. \tag{2.14}$$

By concavity of u and $\int_0^\eta u(s)ds < 0$, it implies that $u(\eta) < 0$.

Hence,

$$u(1) = \alpha \int_0^\eta u(s)ds \geqslant \frac{\alpha\eta}{2}u(\eta) > \frac{u(\eta)}{\eta}, \tag{2.15}$$

which contradicts the concavity of u. □

Lemma 2.3. *Let* $\alpha\eta^2 > 2$. *If* $y \in C(0,1)$ *and* $y(t) \geqslant 0$ *for* $t \in (0,1)$, *then* (2.1)-(2.2) *has no positive solution.*

Proof. Assume (2.1)-(2.2) has a positive solution u.

If $u(1) > 0$, then $\int_0^\eta u(s)ds > 0$, it implies that $u(\eta) > 0$ and

$$\frac{u(1)}{1} = \alpha \int_0^\eta u(s)ds \geqslant \frac{\alpha\eta}{2}u(\eta) = \frac{\alpha\eta^2}{2}\frac{u(\eta)}{\eta} > \frac{u(\eta)}{\eta}, \tag{2.16}$$

which contradicts the concavity of u.

If $u(1) = 0$, then $\int_0^\eta u(s)ds = 0$, this is $u(t) \equiv 0$ for all $t \in [0,\eta]$. If there exists $\tau \in (\eta, 1)$ such that $u(\tau) > 0$, then $u(0) = u(\eta) < u(\tau)$, which contradicts the concavity of u. Therefore, no positive solutions exist. □

In the rest of the paper, we assume that $0 < \alpha\eta^2 < 2$. Moreover, we will work in the Banach space $C[0,1]$, and only the sup norm is used.

Lemma 2.4. *Let* $0 < \alpha < 2/\eta^2$. *If* $y \in C[0,1]$ *and* $y \geqslant 0$, *then the unique solution* u *of the problem* (2.1)-(2.2) *satisfies*

$$\inf_{t\in[\eta,1]} u(t) \geqslant \gamma \|u\|, \tag{2.17}$$

where

$$\gamma := \min\left\{\eta, \frac{\alpha\eta^2}{2}, \frac{\alpha\eta(1-\eta)}{2-\alpha\eta^2}\right\}. \tag{2.18}$$

Proof. Set $u(\tau) = \|u\|$. We divide the proof into three cases.

Case 1. If $\eta \leqslant \tau \leqslant 1$ and $\inf_{t\in[\eta,1]} u(t) = u(\eta)$, then the concavity of u implies that

$$\frac{u(\eta)}{\eta} \geqslant \frac{u(\tau)}{\tau} \geqslant u(\tau). \tag{2.19}$$

Thus,

$$\inf_{t\in[\eta,1]} u(t) \geqslant \eta \|u\|. \tag{2.20}$$

Case 2. If $\eta \leqslant \tau \leqslant 1$ and $\inf_{t\in[\eta,1]} u(t) = u(1)$, then (2.2), (2.13), and the concavity of u implies

$$u(1) = \alpha \int_0^\eta u(s)ds \geqslant \frac{\alpha\eta^2}{2}\left[\frac{u(\eta)}{\eta}\right] \geqslant \frac{\alpha\eta^2}{2}\frac{u(\tau)}{\tau} \geqslant \frac{\alpha\eta^2}{2}u(\tau). \tag{2.21}$$

Therefore,

$$\inf_{t\in[\eta,1]} u(t) \geqslant \frac{\alpha\eta^2}{2}\|u\|. \tag{2.22}$$

Case 3. If $\tau \leqslant \eta < 1$, then $\inf_{t\in[\eta,1]} u(t) = u(1)$. Using the concavity of u and (2.2), (2.13), we have

$$\begin{aligned} u(\sigma) &\leqslant u(1) + \frac{u(1) - u(\eta)}{1-\eta}(0-1) \\ &\leqslant u(1)\left[1 - \frac{1-2/\alpha\eta}{1-\eta}\right] \\ &= u(1)\frac{2-\alpha\eta^2}{\alpha\eta(1-\eta)}. \end{aligned} \tag{2.23}$$

This implies that

$$\inf_{t\in[\eta,1]} u(t) \geqslant \frac{\alpha\eta(1-\eta)}{2-\alpha\eta^2}\|u\|. \tag{2.24}$$

This completes the proof. □

3. Main Results

Now we are in the position to establish the main result.

Theorem 3.1. *Assume* $(H1)$ *and* $(H2)$ *hold. Then the problem* (1.2)-(1.3) *has at least one positive solution in the case*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear), or*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. It is known that $0 < \alpha < 2/\eta^2$. From Lemma 2.1, u is a solution to the boundary value problem (1.2)-(1.3) if and only if u is a fixed point of operator A, where A is defined by

$$\begin{aligned} Au(t) = {} & \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\ & - \int_0^t (t-s)a(s)f(u(s))ds. \end{aligned} \tag{3.1}$$

Denote that

$$K = \left\{ u \mid u \in C[0,1],\ u \geqslant 0,\ \inf_{\eta\leqslant t\leqslant 1} u(t) \geqslant \gamma\|u\| \right\}, \tag{3.2}$$

where γ is defined in (2.18).

It is obvious that K is a cone in $C[0,1]$. Moreover, by Lemmas 2.2 and 2.4, $AK \subset K$. It is also easy to check that $A : K \to K$ is completely continuous.

Superlinear Case ($f_0 = 0$ and $f_\infty = \infty$).

Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leqslant \epsilon u$, for $0 < u \leqslant H_1$, where $\epsilon > 0$ satisfies

$$\frac{2\epsilon}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)ds \leqslant 1. \tag{3.3}$$

Thus, if we let

$$\Omega_1 = \{u \in C[0,1] \mid \|u\| < H_1\}, \tag{3.4}$$

then, for $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned} Au(t) &\leqslant \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))ds \\ &\leqslant \frac{2t\epsilon}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)u(s)ds \\ &\leqslant \frac{2\epsilon}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)ds\|u\| \\ &\leqslant \|u\|. \end{aligned} \tag{3.5}$$

Thus $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \geqslant \rho u$, for $u \geqslant \widehat{H}_2$, where $\rho > 0$ is chosen so that

$$\rho\gamma\frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)ds \geqslant 1. \tag{3.6}$$

Let $H_2 = \max\{2H_1, \widehat{H}_2/\gamma\}$ and $\Omega_2 = \{u \in C[0,1] \mid \|u\| < H_2\}$. Then $u \in K \cap \partial\Omega_2$ implies that

$$\inf_{\eta\leqslant t\leqslant 1} u(t) \geqslant \gamma\|u\| = \gamma H_2 \geqslant \widehat{H}_2, \tag{3.7}$$

and so

$$\begin{aligned}
Au(\eta) &= \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))ds - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\
&\quad - \int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&= \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))ds - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta \left(\eta^2-2\eta s+s^2\right)a(s)f(u(s))ds \\
&\quad - \frac{1}{2-\alpha\eta^2}\int_0^\eta \left(2-\alpha\eta^2\right)(\eta-s)a(s)f(u(s))ds \\
&= \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))ds + \frac{\alpha\eta^2}{2-\alpha\eta^2}\int_0^\eta sa(s)f(u(s))ds \\
&\quad - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta s^2a(s)f(u(s))ds - \frac{2\eta}{2-\alpha\eta^2}\int_0^\eta a(s)f(u(s))ds \\
&\quad + \frac{2}{2-\alpha\eta^2}\int_0^\eta sa(s)f(u(s))ds \\
&= \frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)f(u(s))ds + \frac{2(1-\eta)}{2-\alpha\eta^2}\int_0^\eta sa(s)f(u(s))ds \\
&\quad + \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta s(\eta-s)a(s)f(u(s))ds \\
&\geqslant \frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)f(u(s))ds \\
&\geqslant \frac{2\eta\rho}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)u(s)ds \geqslant \frac{2\eta\rho\gamma}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)ds\|u\| \geqslant \|u\|.
\end{aligned} \tag{3.8}$$

Hence, $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$. By the first past of Theorem 1.1, A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $H_1 \leqslant \|u\| \leqslant H_2$.

Sublinear Case ($f_0 = \infty$ and $f_\infty = 0$).

Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(u) \geqslant Mu$ for $0 < u \leqslant H_3$, where $M > 0$ satisfies

$$\frac{2\eta\gamma M}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)ds \geqslant 1. \tag{3.9}$$

Let

$$\Omega_3 = \{u \in C[0,1] \mid \|u\| < H_3\}, \tag{3.10}$$

then for $u \in K \cap \partial\Omega_3$, we get

$$\begin{aligned}
Au(\eta) &= \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))ds - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\
&\quad - \int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&\geqslant \frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)f(u(s))ds \\
&\geqslant \frac{2\eta\gamma M}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)ds\|u\| \geqslant \|u\|.
\end{aligned} \tag{3.11}$$

Thus, $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_3$. Now, since $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ so that $f(u) \leqslant \lambda u$ for $u \geqslant \widehat{H}_4$, where $\lambda > 0$ satisfies

$$\frac{2\lambda}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)ds \leqslant 1. \tag{3.12}$$

Choose $H_4 = \max\{2H_3, \widehat{H}_4/\gamma\}$. Let

$$\Omega_4 = \{u \in C[0,1] \mid \|u\| < H_4\}, \tag{3.13}$$

then $u \in K \cap \partial\Omega_4$ implies that

$$\inf_{\eta\leqslant t\leqslant 1} u(t) \geqslant \gamma\|u\| = \gamma H_4 \geqslant \widehat{H}_4. \tag{3.14}$$

Therefore,

$$\begin{aligned} Au(t) &= \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\ &\quad - \int_0^t (t-s)a(s)f(u(s))ds \\ &\leqslant \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))ds \\ &\leqslant \frac{2\lambda\|u\|}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)ds \leqslant \|u\|. \end{aligned} \tag{3.15}$$

Thus $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_4$. By the second part of Theorem 1.1, A has a fixed point u in $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$, such that $H_3 \leqslant \|u\| \leqslant H_4$. This completes the sublinear part of the theorem. Therefore, the problem (1.2)-(1.3) has at least one positive solution. □

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